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SOME NEW FIXED POINT RESULTS IN ULTRA METRIC SPACE

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Abstract. The purpose of the present paper is to continue the study of fixed point theory in ultra metric spaces. Concretely, we apply the strong quasi contractive mapping to the results of Gajic [4] and establish some new fixed point results in spherically complete ultrametric space for single valued and multivalued maps and extend the corresponding results for the pair of Junck type mappings. The presented results unify, extend, and improve several results in the related literature.

Keywords: ultrametric space; fixed point, spherically completeness, quasi-contraction

AMS Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

In 1922, Banach established the most famous fundamental fixed point theorem (the so-called Banach contraction principle [1]) which has played an important role in various fields of applied mathematical analysis. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle. Ciric [2], introduced quasi contraction, which is a generalization of Banach contraction principle. Roovij in [10] introduced the concept of Ultrametric space. Later, Petalas et al., [9] and Gajic [4] studied fixed point theorems of contractive type maps on a spherically complete ultra metric spaces [6] which are generalizations of the Banach fixed point theorems. Rao etal.,[11] obtained two coincidence point theorems for three and four self maps in ultra metric space. Kubiaczyk et al. [7] extend the fixed point theorems from the single-valued maps to the set- valued contractive maps. Then Gajic [5] gave some generalizations of the result of [10]. Again, Rao [12] proved some common fixed point theorems [3] for a pair of maps of Jungck type on a spherically complete ultra metric space. Zhang et al. [13] introduced generalized weak-contraction, which is a generalization of Banach contraction principle. Recently, Pant [8] obtained some new fixed point theorems for set-valued contractive and nonexpansive mappings in the setting of ultrametric spaces.

Definition 1.1. [10] *Let (X, d) be a metric space. If the metric d satisfiesis strong triangle inequality:*

for all $x, y, z \in X$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\};$$

Then it is called an ultrametric on X. The pair (X, d) is called an ultrametric space.

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Definition 1.2. [2] *A self mapping $T : X \to X$ on the metric space (X, d) is said to be quasi-contraction if,*

$$d(Tx, Ty) \leq k. \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(x, Ty)\},$$

where $0 \leq k < 1$.

Definition 1.3. [10] *An ultrametric space is said to be spherically complete if the intersection of nested balls in X is non-empty.*

Gajic [4] proved the following result.

Theorem 1.1. *Let (X, d) be a spherically complete ultra metric space. If $T : X \to X$ is a mapping such that*

$$d(Tx, Ty) < max\{d(x, y), d(x, Tx), d(y, Ty)\} \ for \ all \ x, y \in X, \ x \neq y.$$

Then T has a unique fixed point in X.

Theorem 1.2. *(Zorn's lemma). Let S be a partially ordered set. If every totally ordered subset of S has an upper bound, then S contains a maximal element.*

Definition 1.4. *An element $x \in X$ is said to be a coincidence point of $f : X \to X$ and $T : X \to 2^X_C$ (where 2^X_C is the space of all nonempty compact subsets of X). If $fx \in Tx$, we denote*

$$C(f, T) = \{x \in X \ / \ fx \in Tx\},$$

the set of coincidence points of f and T.

Definition 1.5. *Let $(X, \ d)$ be an ultrametric space, and $f : \ X \to X$ and $T : X \to 2^X_C$. f and T are said to be coincidentally commuting at $z \in X$ if $fz \in Tz$ implies $fTz \subseteq Tfz$.*

Definition 1.6. *For A, $B \in B(X)$, ($B(X)$ is the space of all nonempty bounded subsets of X), the Hausdorff metric is defined as:*

$$H(A, B) = \max\left\{\sup_{x\in B} d(x, A), \sup_{y\in A} d(x, b)\right\};$$

where $d(x, A) = \inf\{d(x, a) \ :a \in A\}$.

2. The results

In this section, we apply the strong quasi-contractive mapping [2] on the results of Gajic [4] and establish some new fixed point results for strong quasi-contractive mapping in spherically complete ultrametric space for single valued and multivalued maps and extend the corresponding results for the pair of Junck type mappings.

Let us prove our first main result.

Theorem 2.1. *Let (X, d) is spherically complete ultrametric space satisfying strong quasi-contractive condition*

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \ for \ all \ x, \ y \in X. \tag{1}$$

Then T has a unique fixed point in X.

Proof. Let $S_a = (a, d(a, Ta))$ is a closed sphere whose centre is a and radius $d(a, Ta)$ for all $a \in X$, $d(a, Ta) > 0$ and let F is the collection of all such spheres on which the partial order is defined like $S_b \subseteq S_a$ iff $S_a S_b$. Let F_1 is totaly ordered subfamily of F. As (X, d) is spherically complete so

$$\bigcap_{S_a \in F_1} S_a = S \neq \phi.$$

Let $b \in S \implies b \in S_a$, as $S_a \in F_1$. Hence, $d(a, b) \leq d(a, Ta)$.

If $a = b$ then $S_a = S_b$. Assume that $a \neq b$.

Let $x \in S_b \implies$

$$\begin{aligned}
d(x, b) &\leq d(b, Tb) \leq max\{d(b, a), d(a, Ta), d(Ta, Tb)\} \\
&\leq max\{d(b, a), d(a, Ta), max\{d(a, b), d(b, Tb), d(a, Ta), d(a, Tb), d(b, Ta)\}\} \text{ using } (1) \\
\text{As } d(a, Tb) &\leq \max\{d(a, b), d(b, Tb)\}, \\
\text{and } d(b, Ta) &\leq \max\{d(b, a), d(a, Ta)\}. \\
\text{Then } d(x, b) &\leq max\{d(b, a), d(a, Ta), max\{d(a, b), d(b, Tb), d(a, Ta), \max\{d(a, b), d(b, Tb)\}, \\
&\quad \max\{d(b, a), d(a, Ta)\}\}\} \leq max\{d(b, a), d(a, Ta)\} = d(a, Ta).
\end{aligned}$$

Now

$$d(x, a) \leq \max\{d(x, b), d(b, a)\} \leq d(a, Ta).$$

So

$$x \in S_a \textit{ implies } S_b \subseteq S_a \textit{ for all } S_a \in F_1.$$

Hence S_b is the upper bound of F for the family F_1. Hence by the Zorn's lemma F has a maximal element S_c for some $c \in X$.

Now, we are going to prove that $Tc = c$. Suppose on contrary that $Tc \neq c$. i.e, $d(c, Tc) > 0$.

$$\begin{aligned}
d(Tc, TTc) &< max\{d(c, Tc), d(c, Tc), d(Tc, TTc), d(c, TTc), d(Tc, Tc)\}. \\
\text{As } d(c, TTc) &\leq \max\{d(c, Tc), d(Tc, TTc) \\
d(Tc, TTc) &< max\{d(c, Tc), d(c, Tc), d(Tc, TTc), \max\{d(c, Tc), d(Tc, TTc)\} \\
&< max\{d(c, Tc), d(Tc, TTc) = d(c, Tc).
\end{aligned}$$

implies that $d(Tc, TTc) < d(c, Tc)$.

Then $y \in S_{Tc}$ implies

$$d(y, Tc) \leq d(Tc, TTc) < d(c, Tc).$$

i.e.

$$d(y, Tc) < d(c, Tc).$$

As

$$d(y, c) \leq \max\{d(y, Tc), d(Tc, c)\} = d(c, Tc),$$

$y \in S_c$ implies that $S_{Tc} S_c$, which is contradiction to the maximality of S_c. Hence $Tc = c$.

For the uniqueness let $Tv = v$ be another fixed point. Then

$$\begin{aligned}
d(c, v) &= d(Tc, Tv) < \max\{d(c, v), d(c, Tc), d(v, Tv), d(c, Tv), d(v, Tc)\} \\
&= d(c, v),
\end{aligned}$$

which is contradiction. □

Example 2.1. *Let $X = (R, d)$ is a discrete metric space which is an ultrametric space defined as*

$$d(x,y) = \begin{cases} 0 & if \quad x = y \\ 1 & if \quad x \neq y \end{cases}$$

Let $Tx = c$, then $Tc = c$ is the fixed point where c is any real constant.

Example 2.2. *Let $X = \{\ a, b, c, d\ \}$ with $d(a,b) = d(c,d) = \frac{1}{2}, d(a,c) = d(a,d) = 1$ then (X, d) is spherically complete ultrametric space. Define $T : X \to X$ by $Ta = a, Tb = a, Tc = a, Td = b$. Then $d(Tc, Td) = d(a,b) = \frac{1}{2} = d(c,d)$, the mapping T satisfies the contractive condition of Theorem 2.1.*

Gajic [5] proved the following theorem for multivaled mappings in the setting of ultra metric spaces.

Theorem 2.2. *Let (X, d) be the spherically complete ultrametric space if $T : X \to 2_C^X$ is such that for any $x, y \in X$, $x \neq y$,*

$$H\left(Tx, Ty\right) < \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Then T has a fixed point. (i.e there exist $x \in X$, such that $x \in Tx$).

Now we will prove a multivalued fixed point theorem using strong quasi-contractive mapping in ultrametric space.

Theorem 2.3. *Let (X, d) is spherically complete ultrametric space if $T : X \to 2_C^X$ is such that for any x,y$\in X$, $x \neq y$, satisfying the following condition.*

$$H(Tx, Ty) < \max\{d(x,y), d(x,Tx), d(y,Ty)\ ,d(x,Ty), d(y,Tx)\}\}. \tag{1}$$

Then T has a unique fixed point in X.

Proof. Let $S_a = (a, d(a, Ta))$ is a closed sphere whose centre is a and radius $d(a,Ta) = \inf_{d \in Ta} d(a,d)$ for all $a \in X$ such that $d\left(a, Ta\right) > 0$. Let F is the collection of all such spheres on which the partial order is defined like $S_b \subseteq S_a$ if $S_a S_b$ and let F_1 is totally ordered subfamily of F. As (X, d) is spherically complete

$$_{S_a \in F_1} S_a = S \neq \phi.$$

Let $b \in S \implies b \in S_a$, as S$_a \in F_1$, hence $d(a,b) \leq d(a,Ta)$. Take $u \in Ta$ such that $d(a,u) = d(a,Ta)$ (it is possible because Ta is non-emty compact set). If $a = b$ then $S_a = S_b$. Assume that $a \neq b$. For $x \in S_b$, implies that

$$\begin{aligned} d(x,b) &\leq d(b,Tb) = \inf_{v \in Tb} d\left(b, v\right) \leq max\{d(b,a), d(a,u), \inf_{v \in Tb} d(u,v)\} \\ &\leq \max\{d(a,Ta), H(Ta,Tb)\} \\ &\leq \max\{d(a,Ta), \max\{d(a,b), d(b,Tb), d(a,Ta), d(a,Tb), d(b,Ta)\}\} \text{ using } (1)\,. \\ \text{As } d(a,Tb) &\leq \max\{d(a,b), d(b,Tb)\} \\ \text{and } d(b,Ta) &\leq \max\{d(b,a), d(a,Ta)\}. \\ d(x,b) &\leq max\{d(b,a), d(a,Ta), max\{d(a,b), d(b,Tb), d(a,Ta), \max\{d(a,b), d(b,Tb)\}, \\ &\quad \max\{d(b,a), d(a,Ta)\}\}\}. \\ &\leq max\{d(b,a), d(a,Ta)\} = d(a,Ta). \end{aligned}$$

$$d(x,b) \leq d(a,Ta).$$

Now

$$d(x,a) \leq \max\{d(x,b), d(b,a)\} \leq d(a,Ta).$$

So $x \in S_a$ and $S_b \subseteq S_a$ for all $S_a \in F_1$. Hence S_b is the upper bound of F for the family F_1. Hence by the Zorn's lemma F has a maximal element S_c for some $c \in X$. we are going to prove that $c \in Tc$. Suppose $c \notin Tc$, then there exists $\overline{c} \in Tc$ such that $d(c,\overline{c}) = d(c,Tc)$. Now

$$\begin{aligned} d(\overline{c},T\overline{c}) &\leq H(Tc,T\overline{c}) < max\{d(c,\overline{c}), d(c,Tc), d(\overline{c},T\overline{c}), d(c,T\overline{c}), d(\overline{c},Tc\} \\ &\leq max\{d(c,\overline{c}), d(c,Tc), d(\overline{c},T\overline{c}), \max\{d(c,\overline{c}), d(\overline{c},T\overline{c})\}, \max\{d(\overline{c},c\}, d(c,Tc)\} \\ &\leq max\{d(c,Tc), d(\overline{c},T\overline{c}) = d(c,Tc). \end{aligned}$$

This implies

$$d(\overline{c},T\overline{c}) \quad < d(c,Tc).$$

Let $y \in S_{\overline{c}}$. This implies that

$$d(y,\overline{c}) \leq d(\overline{c},T\overline{c}) < d(c,Tc).$$

As

$$d(y,c) \leq \max\{d(y,\overline{c}), d(\overline{c},c)\} = d(c,Tc),$$

$y \in S_c$ implies that $S_{\overline{c}}S_c$, as $c \notin S_{\overline{c}}$ which is contradiction to the maximality of S_c. Hence $c \in Tc$. □

Example 2.3. *Let $X = \{a,b,c,e\}$, $d(a,c) = d(a,e) = d(b,c) = d(b,e) = 1$, $d(a,b) = d(c,e) = \frac{3}{4}$. It is well known that (X,d) is complete Ultrametric space.*
Define function $T(a) = T(b) = T(c) = \{a\}$ and $T(e) = \{a,b\}$.
Hence, all the conditions of Theorem 2.3 can be satisfied.

Now we extend this idea to pair of Junck type mapping using strong quasi -contractive mapping.

Theorem 2.4. *Let (X,d) be a complete ultrametric space and T, $f : X \to X$ be two self maps on X which satisfies the following conditions:*
1) $TX \subset fX$;
2) $d(Tx,Ty) < max\{d(fx,fy), d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}$ for all x, $y \in X$;
3) fX is spherically complete.
Then T and f have a common point $c \in X$ and if T and f are coincidently commuting at c i.e. $Tfc = fTc$ then $Tc = fc = c$.

Proof. Let $S_a = (fa, d(fa,Ta)) \bigcap fX$ is a closed sphere in fX whose centre is fa for all a in X with radii $d(fa,Ta)$. Let F is the collection of all such spheres on which the partial order is defined like S_aS_b if $S_b \subseteq S_a$. Let F_1 is totaly ordered subfamily of F. As fX is spherically complete

$$_{S_a \in F_1} S_a = S \neq \phi.$$

For $fb \in S \implies fb \in S_a$ as $S_a \in F_1$, hence $d(fa, fb) \leq d(fa, Ta)$. If $fa = fb$ then $S_a = S_b$. Assume that $a \neq b$, and let $x \in S_b$. Then

$$\begin{aligned}
d(x, fb) &\leq d(fb, Tb) \leq max\{d(fb, fa), d(fa, Ta), d(Ta, Tb)\} \\
&\leq max\{d(fb, fa), d(fa, Ta)\ max\{d(fa, fb), d(fb, Tb), d(fa, Ta), \\
&\quad d(fa, Tb), d(fb, Ta)\}\}. \\
\text{As} \quad & d(fa, Tb) \leq \max\{d(fa, fb), d(fb, Tb)\}, \\
\text{and } d(fb, Ta) &\leq \max\{d(fb, fa), d(fa, Ta)\}. \\
\text{Then } d(x, fb) &\leq max\{d(fb, fa), d(fa, Ta)\ max\{d(fa, fb), d(fb, Tb), d(fa, Ta), \\
&\quad \max\{d(fa, fb), d(fb, Tb)\}\}, \\
&\quad \max\{d(fb, fa), d(fa, Ta)\} = d(fa, Ta).
\end{aligned}$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\} \leq d(fa, Ta).$$

implies $x \in S_a$ so $S_b \subseteq S_a$ for all $S_a \in F_1$. Hence S_b is the upper bound of F for the family F_1, hence by the Zorn's lemma F has a maximal element S_c for some $c \in X$. We are going to prove that $Tc = fc$. Suppose $Tc \neq fc$, as $TX \subseteq fX$ there is $e \in X$ such that $Tc = fe$ and $c \neq e$.

$$\begin{aligned}
d(fe, Te) &= d(Tc, Te) < max\{d(fc, fe), d(fc, Tc), d(fe, Te), d\left(fc, Te\right), d\left(fe, Tc\right)\} \\
&= max\{d(fc, fe), d(fe, Te), d\left(fc, Te\right) \\
&\leq max\{d(fc, fe), d(fe, Te), d\left(fe, fe\right), d\left(fc, fe\right), d\left(fe, Te\right)\} \\
&= d(fc, fe).
\end{aligned}$$

This means that

$$d(fe, Te) < d(fc, fe).$$

Let $y \in S_e$. Then

$$d(y, fe) \leq d(fe, Te) < d(fc, Tc),$$

implies

$$d(y, fe) < d(fc, Tc).$$

As

$$d(y, fc) \leq \max\{d(y, fe), d(fe, fc)\} = d(fc, Tc),$$

$y \in S_c$ so $S_e \subseteq S_c$. As $fc \notin S_e \Rightarrow S_e S_c$. It is contradiction to the maximality of S_c, hence $Tc = fc$. Suppose f and T are coincidently commuting at $c \in X$, then $f^2c = f(fc) = fTc = Tfc = T^2c$. Let $fc \neq c$. Now

$$\begin{aligned}
d(Tfc, Tc) &< max\{d(f^2c, fc), d(f^2c, Tc), d(fc, Tc), d\left(f^2c, Tc\right), d\left(fc, Tfc\right)\} \\
&\leq \max\{d(f^2c, Tc), d\left(fc, Tfc\right)\} \\
&= d(Tfc, Tc) \Rightarrow d(Tfc, Tc) < d(Tfc, Tc).
\end{aligned}$$

Which is contradiction, hence $Tc = fc = c$.

For the uniquness, let e be another fixed point. i.e, $Te = fe = e$.

$$d(Te, Tc) < \max\{d(fe, fc), d(fe, Te), d(fc, Tc), d\left(fe, Tc\right), d\left(fc, Te\right)\} < d(Te, Tc),$$

which is contradiction. □

Example 2.4. *Let $(X = R, d)$ is discrete matric space,*

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Let $Tx = 7$, $fx = \frac{x+7}{7}$ satisfy the above conditions, has a fixed point $x = 7$.

Remark 2.1. *We can suppose that f is spherically complete, if f is surjective.*

Now we prove the result for Junck type multivalued functions.

Theorem 2.5. *Let (X, d) be an ultrametric space and Let $T : X \to 2^X_C$ and f is self map on X which satisfy the following conditions;*

i) $Tx \subseteq fX, for\ all\ x \in X$;

ii) $H(Tx, Ty) < max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}$ for all x, $y \in X$, $x \neq y$;

iii) fX is spherically complete.

Then there exists $z \in X$ such that $fz \in Tz$.

Further assume that

iv) $d(fx, fu) \leq H(Tfy, Tu)$ for all x, y, $u \in X$ with $fx \in Ty$ and

f and T are coincidentally commuting at c , then fc is the unique common fixed point of f and T.

Proof. Let $S_a = (fa, d(fa, Ta)) \bigcap fX$ is a closed sphere in fX whose centre is fa and radius $d(fa, Ta) = \inf_{d \in Ta} d(fa, d)$ for all $a \in X$, $d\,(fa, Ta) > 0$. Let F is the collection of all such spheres on which the partial order is defined like $S_b \subseteq S_a$ if $S_a S_b$ and let F_1 is totaly ordered subfamily of F. As (X, d) is spherically complete so

$$\bigcap_{S_a \in F_1} S_a = S \neq \phi.$$

Now $fb \in S \implies fb \in S_a$, as $S_a \in F_1$, hence $d(fa, fb) \leq d(a, Ta)$.

Take $u \in Ta$ such that $d(fa, u) = d(fa, Ta)$ (it is possible because Ta is non-empty compact set). If $fa = fb$ then $S_a = S_b$. Assume that $fa \neq fb$.

For $x \in S_b$ implies

$$\begin{aligned} d(x, fb) &\leq d(fb, Tb) = \inf_{v \in Tb} d\,(fb, v) \leq max\{d(fb, fa), d(fa, u), \inf_{v \in Tb} d(u, v)\}. \\ d\,(fb, fa) &\leq \max d(fa, Ta), H(Ta, Tb) \leq max\{d(fb, fa), d(fa, Ta) \\ &\quad max\{d(fa, fb), d(fb, Tb), d(fa, Ta), d(fa, Tb), d(fb, Ta)\}\}. \\ \text{As } d(fa, Tb) &\leq \max\{d(fa, fb), d(fb, Tb)\}, \\ \text{and } d(fb, Ta) &\leq \max\{d(fb, fa), d(fa, Ta)\}. \\ d\,(x,\ fb) &\leq max\{d(fb, fa), d(fa, Ta),\ max\{d(fa, fb), d(fb, Tb), d(fa, Ta), \\ &\quad \max\{d(fa, fb), d(fb, Tb), d(fb, fa), d(fa, Ta)\}\}\} = d(fa, Ta). \end{aligned}$$

Now

$$d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\} \leq d(fa, Ta),$$

implies

$$x \in S_a \text{ so } S_b \subseteq S_a \ for\ all\ S_a \in F_1.$$

Hence S_b is the upper bound of F for the family F_1, hence by the Zorn's lemma F has a maximal element S_c for some $c \in X$. Now, we are going to prove that $fc \in Tc$.

Suppose $fc \notin Tc$, then there exists $f\overline{c} \in Tc$ such that $d(fc, f\overline{c}) = d(fc, Tc) > 0$. Then

$$\begin{aligned} d(f\overline{c}, T\overline{c}) &\leq H(Tc, T\overline{c}) < max\{d(fc, f\overline{c}), d(fc, Tc), d(f\overline{c}, T\overline{c}), d(fc, T\overline{c}), d(f\overline{c}, Tc)\} \\ &\leq max\{d(fc, Tc), d(f\overline{c}, T\overline{c}), \max\{d(fc, f\overline{c}), d(f\overline{c}, T\overline{c}), \max\{d(fc, f\overline{c}), d(fc, Tc)\}\} \\ &= d(fc, Tc). \end{aligned}$$

This implies

$$d(f\overline{c}, T\overline{c}) \quad < d(fc, Tc).$$

Let $y \in S_{\overline{c}}$. Then $d(y, f\overline{c}) \leq d(f\overline{c}, T\overline{c}) < d(fc, Tc)$.
Since

$$d(y, fc) \leq \max\{d(y, f\overline{c}), d(f\overline{c}, fc)\} = d(fc, Tc).$$

$y \in S_c$ so $S_{\overline{c}} \subseteq S_c$. As $fc \notin S_{\overline{c}}$, it is contradtion to the maximality of S_c, hence $fc \in Tc$.
Further assume (iv) and write $fc = e$ .Then $e \in Tc$.

$$d(e, fe) = d(fc, fe) \leq H(Tfc, Te) = H(Te, Te) = 0.$$

This implies that $fe = e$. From (iv), $e = fe \in fTc \subseteq Tfz = Te$. Thus $fc = e$ is a common fixed point of f and T.

Suppose $h \in X$, such that $e \neq h = fh \in Th$. From (iv)

$$\begin{aligned} d(e, h) &= d(fe, fh) \leq H(Tfe, Th) = H(Te, Th) \\ &< max\{d(fe, fh), d(fe; Te), d(fh, Th), d(fe, Th), d(h, Te)\} = d(e, h). \end{aligned}$$

This implies that $e = h$. Thus $e = fc$ is the unique common fixed point of f and T. □

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4. Conflict of interests

The authors declare that they have no competing interests.

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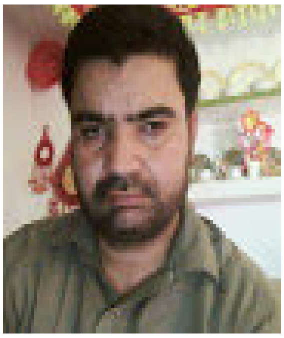

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